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APPLICATION OF THE SELF-SIMILAR INTERPOLATION METHOD TO PROBLEMS OF RAREFIED GAS DYNAMICS†

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The effectiveness of the method of self-similar interpolation [1] in its simplest version is demonstrated by solving problems of slow plane Couette and Poiseuille flows of a rarefied gas and the problem of the structure of a strong shock wave in a monatomic gas. Interpolations of the function with respect to its specified asymptotic representations of a different form at the ends of the interval in which the function is specified, usually semi-infinite, are obtained. © 2005 Elsevier Ltd. All rights reserved.

1. THE METHOD OF SELF-SIMILAR INTERPOLATION

Suppose we know the following asymptotic forms for the required function $f(x)$, $x = [0, \infty)$

$$f(x) = \begin{cases} b, & x \to 0 \\ \sum_{i=0}^{N} A_i x^{\alpha_i}, & x \to \infty \end{cases} \tag{1.1}$$

Using the method of self-similar interpolation [1], we construct interpolation formulae of different orders. The asymptotic form for $x \to 0$ is fixed, and hence we will only give approximations for $f(x \to \infty)$.

The first order. We will use the principal terms of the expansion

$$f(x) = A_0 x^{\alpha_0}, \quad x \to \infty \tag{1.2}$$

The interpolation formula has the form

$$f^*(x) = (b^{1/n} + Bx)^n \tag{1.3}$$

Quantities obtained by this method are denoted by an asterisk.

As $x \to 0$ we have $f^*(x) \to f(x)$. The unknowns B and n are found from the equation

$$B^n x^n = A_0 x^{\alpha_0}$$

which follows from relations (1.2) and (1.3) as $x \to \infty$. Hence $n = \alpha_0$, $B = A_0^{1/\alpha_0}$. As a result we obtain the formula

$$f^*(x) = (b^{1/\alpha_0} + A_0^{1/\alpha_0} x)^{\alpha_0} \tag{1.4}$$

which gives the correct asymptotic forms both as $x \to 0$ and as $x \to \infty$.

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The second order. We have

$$f(x) = A_0 x^{\alpha_0} + A_1 x^{\alpha_1}, \quad x \to \infty \tag{1.5}$$

$$f^*(x) = [(b^{1/n} + Cx)^{n/m} + Dx^2]^m \tag{1.6}$$

To determine n, m, C and D we will consider the asymptotic form as $x \to \infty$. Initially we equate the principal terms in formulae (1.6) and (1.5); as a result we obtain the equation

$$D^m x^{2m} = A_0 x^{\alpha_0}$$

Hence $m = \alpha_0/2$, $D = A_0^{2/\alpha_0}$. Henceforth, in Eq. (1.6), we will neglect the term $b^{1/n}$ compared with Cx and linearize. We obtain

$$n = \frac{\alpha_0}{2}\beta, \quad C = \left(\frac{2}{\alpha_0} A_0^{\gamma} A_1\right)^{1/\beta}, \quad \beta = \alpha_1 - \alpha_0 + 2, \quad \gamma = \frac{2}{\alpha_0} - 1$$

The third order. We have

$$f(x) = A_0 x^{\alpha_0} + A_1 x^{\alpha_1} + A_2 x^{\alpha_2}, \quad x \to \infty$$

$$f^*(x) = \{[(b^{1/n} + Dx)^{n/m} + Ex^2]^{m/\rho} + Gx^3\}^{\rho}$$

In practical problems, as a rule, we are given a few terms of the asymptotic series at the ends of the interval. In order to obtain formulae of the form (1.1), we will use algebraic transformations (addition or subtraction of constants, and multiplication or division by certain functions of x, as, for example, in Section 2), if in the asymptotic series there is a term with a logarithm, an exponential transformation is used with further expansion of the exponential function in a power series as $x \to 0$ (Section 3), and if there is a term with an exponential function in the asymptotic series, a logarithmic transformation is carried out (Section 4). We then find an interpolation formula for the converted expressions and carry out an inverse transformation.

2. THE COUETTE PROBLEM

Consider the slow flow (local Mach number $M \ll 1$) of a rarefied gas between parallel plates, which move relative to one another with equal and opposite velocities. The integral equation for the velocity profile has the form [2]

$$g(x) = f^-(x) + \frac{\alpha}{\sqrt{\pi}} \int_{-1/2}^{1/2} T_{-1}(|x-s|) g(s) ds \tag{2.1}$$

$$f^{\pm}(x) = \frac{1}{\sqrt{\pi}}\left\{T_0\left(\alpha\left(\frac{1}{2} - x\right)\right) \pm T_0\left(\alpha\left(\frac{1}{2} + x\right)\right)\right\}$$
$$T_i(x) = \int_0^{\infty} z^i \exp\left(-z^2 - \frac{x}{z}\right) dz, \quad i = 0, -1; \ \alpha = \frac{1}{\mathrm{Kn}} \tag{2.2}$$

It is required to obtain the friction stress P_{xz}. When $\alpha \to 0$ we have [2]

$$P = P_{xz}/P_{xz}^0 = 1 - \sqrt{\pi}\alpha/2 \tag{2.3}$$

When $\alpha \to \infty$ (taking slippage into account)

$$P = \sqrt{\pi}/\alpha - 2\sqrt{\pi}/\alpha^2 \tag{2.4}$$

where P_{xz}^0 is the friction stress in the free-molecule case.

The self-similar interpolations of the different orders have the following form the first order

$$P = \begin{cases} 1, & \alpha \to 0 \\ \sqrt{\pi}/\alpha, & \alpha \to \infty \end{cases}; \quad P^* = \frac{\sqrt{\pi}}{\sqrt{\pi}+\alpha} \tag{2.5}$$

the second order

$$P = \begin{cases} 1-\sqrt{\pi}\alpha/2, & \alpha \to 0 \\ \sqrt{\pi}/\alpha, & \alpha \to \infty \end{cases}; \quad P^* = \frac{\sqrt{\pi}\alpha}{2}\left[1+\frac{\pi\alpha}{4}(\alpha+2\sqrt{\pi})\right]^{-1/2} \tag{2.6}$$

the third order

$$\begin{gathered} P = \begin{cases} 1-\sqrt{\pi}\alpha/2, & \alpha \to 0 \\ \sqrt{\pi}/\alpha - 2\sqrt{\pi}/\alpha^2, & \alpha \to \infty \end{cases} \\ P^* = \frac{\sqrt{\pi}\alpha}{2}\left[1+\frac{\pi\alpha}{8}(6\pi(\sqrt{\pi}-1)+3\pi\alpha+\sqrt{\pi}\alpha^2)\right]^{-1/3} \end{gathered} \tag{2.7}$$

Equation (2.1) was solved numerically for several values of α by the variational method of least squares. We chose a twenty-degree polynomial as the test function.

A comparison of the values of the ratio $P = P_{xz}/P_{xz}^0$ as function of the inverse Knudsen number α (N are the results of a numerical calculation, and I_1, I_2 and I_3 are the interpolations of the first, second and third orders (formulae (2.5)–(2.7)) is given below

α	0.1	0.3	1	3	10	30	100
N	0.927	0.819	0.600	0.360	0.147	0.0559	0.0176
I_1	0.947	0.855	0.639	0.371	0.150	0.0558	0.0174
I_2	0.922	0.807	0.585	0.344	0.145	0.0549	0.0173
I_3	0.924	0.817	0.603	0.356	0.147	0.0554	0.0174

The maximum interpolation error is 6% (for $\alpha = 1$), 2.7% (for $\alpha = 2$) and 0.6% (for $\alpha = 30$) for interpolation of the first, second and third orders respectively.

3. POISEUILLE FLOW

Consider the slow flow of a rarefied gas between two infinite parallel fixed plates under the action of a small pressure gradient $\partial p/\partial x = -Kp_0$, $K = \text{const}$, where p_0 is the pressure at $x = 0$, and the x coordinate is directed along the plates and is related to the distance between them. The integral equation for the velocity profile of the gas $u_x = g(y)V$, $V^2 = 2kT/m$ has the form [2, 3]

$$g(x) = \frac{K}{2\alpha}[1-f^+(x)] + \frac{\alpha}{\sqrt{\pi}}\int_{-1/2}^{1/2} T_{-1}(\alpha|x-s|)g(s)ds \tag{3.1}$$

The functions f^+ and T_{-1} and the quantity α are given by formulae (2.2).

It is required to calculate the dimensionless volume flow rate of the gas

$$Q(\alpha) = \int_{-1/2}^{1/2} g(x)dx$$

The asymptotic representations of the function $Q(\alpha)$ have the form [2, 3] (γ is Euler's constant)

$$Q(\alpha) = \begin{cases} -\ln\alpha/\sqrt{\pi} + (1-\gamma/2)/\sqrt{\pi}, & \alpha \to 0 \\ \alpha/6 + 1.0162, & \alpha \to \infty \end{cases} \tag{3.2}$$

The self-similar interpolation is given by the formula

$$Q^*(\alpha) = A + B\alpha + C\ln(D + E\alpha^{-1}); \quad A = 1.0162$$
$$B = 1/6, \quad C = 1/\sqrt{\pi}, \quad D = 3/2, \quad E = 0.3363 \tag{3.3}$$

The function $Q(\alpha)$ has a minimum (the well-known Knudsen paradox [2]); when $\alpha \to 0$ the function $Q(x)$ contains a logarithmic term. These features complicate the determination of the interpolation (compared with the case considered in Section 2). Nevertheless, the results of calculations using formula (3.3) differ only slightly from the numerical results in [3] (see below, where N are the results of numerical calculations of $Q(\alpha)$ [3] and I are the results of using formula (3.3))

α	0.01	0.05	0.1	0.5	1	5	10	50	100	500
N	3.050	2.302	2.033	1.618	1.539	1.991	2.768	9.370	17.70	84:36
I	3.044	2.346	2.066	1.560	1.470	1.923	2.721	9.357	17.69	84.35

The maximum difference is equal to 4.5% for $\alpha = 1$, in the region of the minimum flow rate.

4. THE STRUCTURE OF A STRONG SHOCK WAVE IN A MONATOMIC GAS

The one-dimensional time-independent motion of a monatomic gas in the direction of the x axis is described using Boltzmann's kinetic equation, while we will represent in the form

$$c_x \partial f(x, \mathbf{c})/\partial x = J_1(x, \mathbf{c}) - f(x, \mathbf{c}) J_2(x, \mathbf{c})$$
$$J_1(x, \mathbf{c}) = \int f(x, \mathbf{c}_+) f(x, \mathbf{c}_-) d\Omega, \quad J_2(x, \mathbf{c}) = \int f(x, \mathbf{c}_1) d\Omega \tag{4.1}$$
$$d\Omega = g b db d\varepsilon d\mathbf{c}_1, \quad \mathbf{c}_\pm = (\mathbf{c} + \mathbf{c}_1 \pm g\mathbf{k})/2$$

Here $c = (c_x, c_y, c_z)$ is the velocity of a molecule, $g = |c - c_1|$ is the modulus of the relative velocity of the molecules on collision, b is the impact parameter, ε is the azimuthal angle and $\mathbf{c}_\pm$, $\mathbf{k}$ are the velocities of the molecules and the unit vector of the relative velocity of the molecules after collision.

The number density, velocity and temperature of the gas are expressed in terms of the distribution function $f(x, c)$ by the formulae

$$n = \int f(x, \mathbf{c}) d\mathbf{c}, \quad u = \frac{1}{n}\int c_x f(x, \mathbf{c}) d\mathbf{c}$$
$$T = \frac{m}{3kn}\int [(c_x - u)^2 + c_y^2 + c_z^2] f(x, \mathbf{c}) d\mathbf{c} \tag{4.2}$$

When $x \to \mp\infty$ the distribution function approaches the Maxwell functions

$$f_j^M = n_j\left(\frac{\theta_j}{\pi}\right)^{3/2} \exp\{-\theta_j[(c_x - u_j)^2 + c_y^2 + c_z^2]\}, \quad \theta_j = \frac{m}{2kT_j}; \quad j = 1, 2 \tag{4.3}$$

The values of the density, velocity and temperature of the gas upstream ($x = -\infty$) and downstream of the shock wave ($x = \infty$) are equal to n_1, u_1 and T_1 and n_2, u_2 and T_2 respectively. These values are connected by the well-known relations [4]

$$\frac{n_2}{n_1} = \frac{u_1}{u_2} = \frac{4M_1}{M_1 + 3}, \quad \frac{T_2}{T_1} = \frac{(M_1 + 3)(5M_1 - 1)}{16M_1} \tag{4.4}$$

where M_1 is the Mach number upstream of the shock.

The integral form of Eq. (4.1) for the problem of the structure of the shock wave has the form [2] ($\hat{A}$ is an operator)

$$f(x, \mathbf{c}) = \hat{A}f \equiv \left\{ \int_{-\infty}^{x} W(\tau, \mathbf{c})d\tau,\ c_x > 0;\ \int_{x}^{\infty} W(\tau, \mathbf{c})d\tau,\ c_x < 0 \right\}$$
$$W(\tau, \mathbf{c}) = \frac{J_1(\tau, \mathbf{c})}{c_x}\exp\left[-\int_{\tau}^{x}\frac{J_2(\tau_1, \mathbf{c})}{c_x}d\tau_1\right] \tag{4.5}$$

We will use the "pseudo-Maxwellian" model of the molecules, when the intermolecular potential is equal to χr^{-4}, where r is the distance between the molecules. The molecules are elastic spheres, the diameter of which d depends on the relative velocity of the molecules on collision [5] and is equal to the distance of closest approach of the molecules on collision, i.e. $(4\chi/(mg^2))^{1/4}$. We will introduce the quantity σ_0 by the equation $\pi d^2 = \sigma_0 g^{-1}$. Then $J_2(x, \mathbf{c}) = \sigma_0 n$.

We will solve integral equation (4.5) by the method of successive approximations $f^{(n)} = \hat{A}f^{(n-1)}$, taking into account boundary conditions (4.3) with the choice of the following zeroth approximation [6]

$$f^{(0)}(x, \mathbf{c}) = \begin{cases} f_1^M & \text{when } -\infty < x < 0 \ \text{ or } \ x = 0,\ c_x > 0 \\ f_2^M & \text{when } 0 < x < \infty \ \text{ or } \ x = 0,\ c_x < 0 \end{cases}$$

The quantities f_1^M and f_2^M are calculated for (n_1, θ_1, u_1) and (n_2, θ_2, u_2) respectively.

Using the equation

$$f_j(x, \mathbf{c}_+)f_j(x, \mathbf{c}_-) = f_j(x, \mathbf{c})f_j(x, \mathbf{c}_1), \quad j = 1, 2$$

we obtain in the first approximation

$$f^{(1)}(x, \mathbf{c}) = \begin{cases} f_1^M + \xi(f_2^M - f_1^M)B_1(x, c_x) & \text{when } x < 0 \\ f_2^M + (1-\xi)(f_1^M - f_2^M)B_2(x, c_x) & \text{when } x > 0 \end{cases} \tag{4.6}$$

$$\xi = \begin{cases} 1 & \text{when } c_x < 0 \\ 0 & \text{when } c_x > 0 \end{cases}, \quad B_j(x, c_x) = \exp(-\sigma_0 n_j x/c_x), \quad j = 1, 2 \tag{4.7}$$

Using relations (4.2) and (4.7), we obtain an expression for the density in the first approximation

$$n = \begin{cases} n_1(1 - n_{11} + n_{12}), & x < 0 \\ n_2(1 - n_{22} + n_{21}), & x > 0 \end{cases} \tag{4.8}$$

Here

$$n_{jm} = \frac{n_m}{n_j}\left(\frac{\theta_m}{\pi}\right)^{1/2} N_{jm}, \quad N_{1m} = \int_{-\infty}^{0} A_m B_1 dc_x, \quad N_{2m} = \int_{0}^{\infty} A_m B_2 dc_x$$
$$A_m = \exp\left[-\theta_m(c_x - u_m)^2\right], \quad m = 1, 2 \tag{4.9}$$

We will consider the asymptotic behaviour of the integral

$$J_i(y, q) = \int_0^{\infty} w^i \exp\left[-(w-q)^2 - \frac{y}{w}\right]dw =$$
$$= y^{(i+1)/3}\int_0^{\infty} z^i \exp[-y^{2/3}\Phi(z)]dz, \quad \Phi(z) = (z - qy^{-1/3})^2 + \frac{1}{z} \tag{4.10}$$

when $y \to \infty$. Here i, y and q are parameters. The variable z is introduced by the formula $w = y^{1/3}z$.

The function $\Phi(z)$ has a minimum at the point z_0, where

$$z_0 = \frac{1}{2^{1/3}} + \frac{1}{3}\varepsilon + \frac{2^{1/3}}{9}\varepsilon^2 + \frac{2^{5/3}}{81}\varepsilon^3 + \ldots, \quad \varepsilon = qy^{-1/3}$$

Following Laplace's method [7], we expand the function $\Phi(z)$ in a Taylor series in the neighbourhood of the point z_0

$$\Phi(z) = \frac{1}{z_0} + (z_0 - \varepsilon)^2 + \left(1 + \frac{1}{z_0^3}\right)(z - z_0)^2 - \frac{(z - z_0)^3}{z_0^4} + \frac{(z - z_0)^4}{z_0^5} + \ldots$$

and substitute it into expression (4.10). We obtain

$$J_i = y^{(i+1)/3} z_0^i \exp\left[-y^{2/3}\left((z_0 - \varepsilon)^2 + \frac{1}{z_0}\right)\right] \int_0^\infty \exp\left[-y^{2/3}\left(1 + \frac{1}{z_0^3}\right)(z - z_0)^2\right] dz \tag{4.11}$$

Evaluating this integral, expanding the expression obtained in series in ε, and replacing ε by $qy^{-1/3}$, we have

$$J_i(y, q)|_{y \to \infty} = \sqrt{\frac{\pi}{3}} \beta^i \left[1 + \frac{i+1}{3} q \beta^{-1}\right] \exp\left[-3\beta^2 + 2q\beta - \frac{2}{3}q^2\right], \quad \beta = \left(\frac{y}{2}\right)^{1/3} \tag{4.12}$$

Hence we obtain the following asymptotic expressions as $x \to \infty$

$$n_{ij} = \frac{1}{\sqrt{3}} \frac{n_j}{n_i} \exp\left[-3\theta_j^{1/3} \left(\frac{\sigma_0 n_i |x|}{2}\right)^{2/3}\right], \quad i, j = 1, 2 \tag{4.13}$$

For large numbers M_1 in formulae (4.8) the quantities n_{11} and n_{12} are small compared with n_{12} and n_{22} respectively. Neglecting them, we obtain

$$n = \begin{cases} n_1(1 + n_{12}), & x \to -\infty \\ n_2(1 - n_{22}), & x \to \infty \end{cases} \tag{4.14}$$

The method of constructing the interpolation formulae differs from those used in Sections 2 and 3 in specifying the asymptotic representations at the ends of the infinite interval. The following method is used. It is assumed that the function $n(x)$ has continuous derivatives up to the third order inclusive, and the function is split into left- and right-hand parts at the point $x = 0$. Interpolations are found for the left- and right-hand parts, containing the unknown values of the function and its first derivative at the point $x = 0$. To determine these values, it is assumed that the second and third derivatives of the left- and right-hand parts are equal at $x = 0$. The required interpolation is given by complex formulae, which are not presented here. The results of calculations of the density profile for $M_1 = 10$–30 were compared with the results of calculations by the standard method for kinetic theory by direct statistical modelling. In Fig. 1, for $M_1 = 11$, we show the interpolation curve of the normalized density n (the

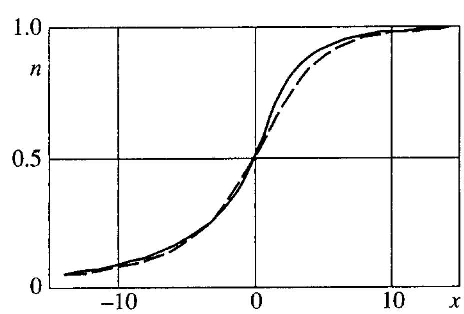

Fig. 1

continuous curve) and the data of a numerical calculation (the dashed curve). The x coordinate refers to the mean free path upstream of the wave. The relative difference between these results does not exceed 10%, which is close to the error of calculations by direct statistical modelling.

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